GEOPHYSICS, VOL. 56, NO. 8 (AUGUST 1991); P. 1179–1184, 8 FIGS., 1 TABLE.

An exact solution for the gravity curvature (Bullard B) correction

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ABSTRACT

The complete Bouguer reduction includes, in addition to the simple Bouguer slab correction (Bullard A), both curvature (Bullard B) and terrain (Bullard C) corrections. A new closed-form formula for the curvature correction is derived for which the calculated values differ from those published by Swick by more than 0.5 mGal for high elevations. These corrections reduce those of an infinite slab (Bullard A) to that of a spherical cap having a surface radius of 166.7 km. The spherical cap produces a lesser effect than the infinite slab because of the "truncation" of that part of the slab above the earth and extending to infinity, but it produces a greater effect than the slab because of subslab earth resulting from curvature. The physical significance of the correction lies in the combination of these two differences, which are each a function of elevation. The Bullard B surface radius (166.7 km: outer radius of the Hayford-Bowie system) is reaffirmed by the exact formula to be appropriate for exploration surveys. Three series approximations are presented and compared, but the exact Bullard B formula is very efficient and easy to program for routine data processing.

INTRODUCTION

The purpose of the Bullard B correction as a step in producing the Bouguer anomaly is to convert the geometry for the Bouguer correction from an infinite slab to a spherical cap (Figure 1) whose thickness is the elevation of the station and whose radius (arc length) from the station is 166.735 km (the outer radius of the Hayford-Bowie terrain system). Schleusener (1953) developed a curvature formula to show the inadequacy of the infinite slab, but failed to recognize the correction procedure previously established by Bullard (1936) and Swick (1942). Curvature corrections are not discussed by Nettleton (1976), Dobrin and Savit (1988), Telford et al. (1976), Parasnis (1986), or Jacobs et al. (1959). A frequently quoted source for this correction is Swick (1942), in which a table derived from Bullard (1936) gives values only to 0.1 mGal. Also, the values are in error (by up to 0.5 mGal) and the elevation sampling is coarse (up to 500 m) and limited to 5000 m. Snyder's (1968) incorrect formula led to erroneous tabulated values for South Park, CO, which are, however, very close to those of Swick. Takin and Talwani (1966) took into account earth sphericity while suggesting a single computation (combining the terrain correction with the simple Bouguer correction) for the complete Bouguer anomaly. Although they refer to Bullard's (1936) paper, they did not propose that the Bullard or Swick use of a curvature correction (i.e., applied to the "Bullard A") be implemented. Heiskanen and Vening Meinesz (1958, p. 155–164) discussed the curvature "b" correction and gave its value (the same as Bullard) for a few elevations up to 1500 m. It is probably not surprising that this correction has been frequently neglected for exploration surveys.

The geophysical exploration community is largely unaware of the Swick/Bullard curvature discrepancy. However, the U.S. Geological Survey recognized through an internal memorandum in the 1960s (Plouff, personal communication) that the Bullard values do not closely agree with Lambert (1930). Lambert presented a closed-form solution to the curvature problem intended for isostatic reductions different in form from the one presented here. Although it is somewhat puzzling that Bullard (1936) chose not to avail himself of the Lambert (1930) formula, using instead numerical values tabulated from Cassinis and others (1937), it is less mysterious that the gravity data reduction community, especially explorationists, would embrace the Bullard approach of defining the Bouguer anomaly. Lambert (1930) did not address the problem from the exploration point of view. The USGS developed a power-series approximation (see

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below) to the Lambert formula and replaced their use of the Swick values with those calculated by the approximation.

THE BULLARD B EXACT FORMULA

I derive here a new version of the curvature formula in closed form (Appendix), which is

$$BB = 2\pi\gamma\rho(\mu h - \lambda R), \quad (1)$$

where BB, γ, ρ, h, and R are the Bullard B correction, the gravitational constant, density, elevation (from sea level), and the earth radius to the station ($R_0 + h$, where R_0 is the earth's radius), respectively; $2\pi\gamma\rho h$ is the familiar simple Bouguer slab formula; μ and λ are dimensionless coefficients defined in the Appendix. Table 1 gives the values for the Bullard B correction to within one microgal for an interval of 100 m for elevations from sea level to 6300 m. These are plotted in Figure 2. The calculations are based on an earth radius of 6371 km and a cap density of 2.67 g/cm^3.

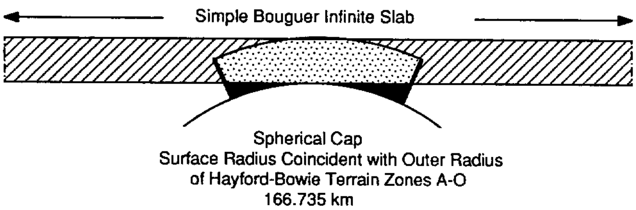

FIG. 1. Geometry of spherical cap in relation to infinite slab. The Bullard B correction modifies the simple Bouguer slab value (Bullard A) to that of a cap having a surface radius of nearly 167 km and a thickness the same as that of the infinite slab (station elevation). This is equivalent to removing all of the slab above the earth's surface and beyond 167 km whether above or below the earth's surface (i.e., all of the slant-shaded zone) and adding the part of the cap below the slab (i.e., the solid black zone). That part of the cap shown in stipple pattern is common to both the cap and the slab and therefore does not enter into the Bullard B correction. The sum of the stipple zone and the black zone constitutes the entire spherical cap. All dimensions are greatly exaggerated to clearly show the nature of the correction.

An examination of Figures 1 and 2 shows that, for low elevations, part of the spherical cap directly underlying the infinite slab (black zone in Figure 1) dominates the correction (i.e., the truncation effect is relatively small). For an elevation of 4150 m, the correction is nearly zero, so that the truncation effect is balanced by the cap material at the base of the infinite slab. For higher elevations, the truncation effect dominates. The physical nature of curvature effects are further discussed by Schleusener (1953) and Heiskanen and Vening Meinesz (1958). The former points out the inadequacy of the simple Bouguer slab, while the latter indicates the need for Bullard's correction.

The difference graph shown in Figure 2 indicates that this correction in high-precision engineering surveys (and other applications) may be as high as 14 microgals per meter of elevation difference at low elevations. This difference is a

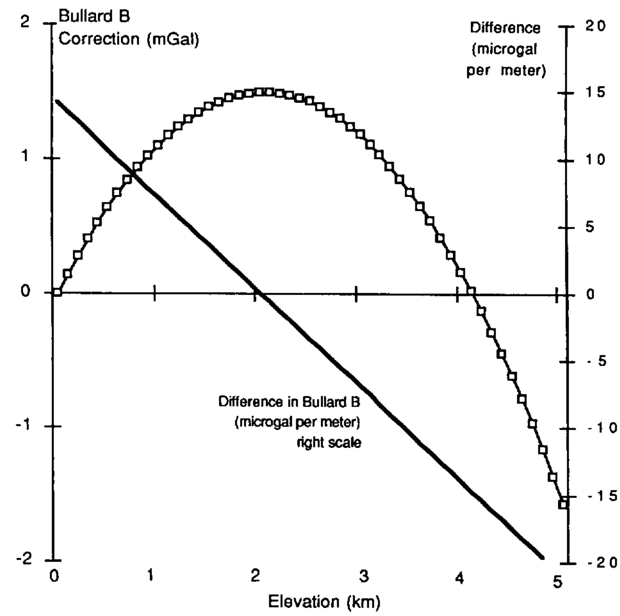

FIG. 2. Bullard B correction and its first difference.

Table 1. Values of the Bullard B correction

h (m)	BB (mGal)	h (m)	BB (mGal)	h (m)	BB (mGal)	h (m)	BB (mGal)
0	0.000	1600	1.439	3200	1.071	4800	−1.099
100	0.143	1700	1.469	3300	0.988	4900	−1.295
200	0.279	1800	1.491	3400	0.898	5000	−1.497
300	0.407	1900	1.507	3500	0.801	5100	−1.707
400	0.529	2000	1.516	3600	0.697	5200	−1.923
500	0.644	2100	1.518	3700	0.586	5300	−2.147
600	0.751	2200	1.512	3800	0.468	5400	−2.377
700	0.852	2300	1.500	3900	0.343	5500	−2.615
800	0.945	2400	1.481	4000	0.211	5600	−2.859
900	1.032	2500	1.454	4100	0.072	5700	−3.111
1000	1.111	2600	1.420	4200	−0.074	5800	−3.370
1100	1.183	2700	1.380	4300	−0.228	5900	−3.635
1200	1.248	2800	1.332	4400	−0.388	6000	−3.908
1300	1.307	2900	1.278	4500	−0.555	6100	−4.188
1400	1.358	3000	1.216	4600	−0.729	6200	−4.475
1500	1.402	3100	1.147	4700	−0.911	6300	−4.768

function of the vertical separation of stations and not of their horizontal separation.

A comparison between the widely used Bullard values tabulated by Swick and others and those calculated by equation (1) is given in Figure 3. The Swick/Bullard values are generally too large and not smooth.

THE BULLARD B SURFACE RADIUS

It is interesting to question the selection of 166.7 km as the surface radius of the spherical cap, as very little can be found in the literature to justify this choice. The selection of 166.735 km (Bullard, 1936) for the spherical cap's surface radius (which is the outer radius of the Hayford-Bowie Zone

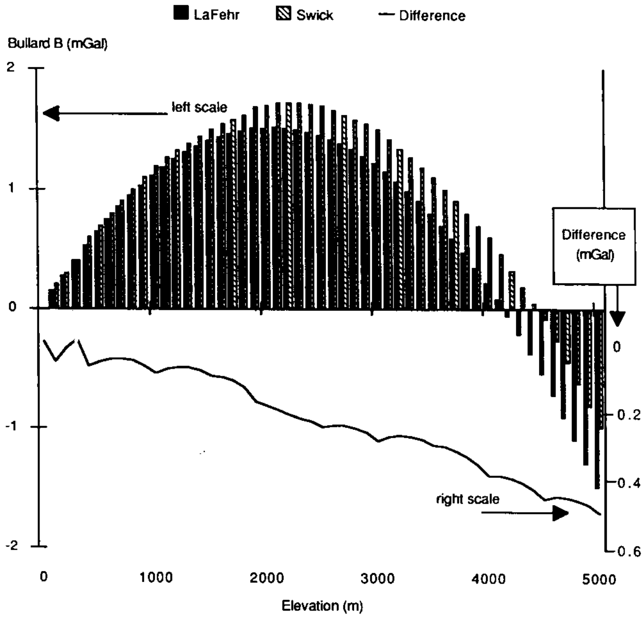

FIG. 3. Comparison between Swick and LaFehr calculations.

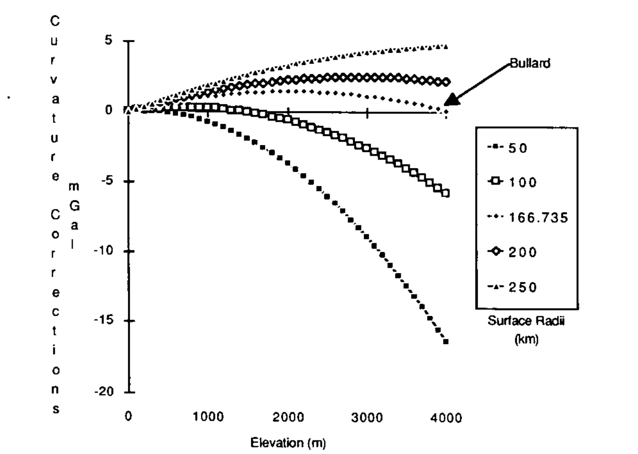

FIG. 4. Curvature corrections for caps of different surface radii.

O) was based on minimizing the difference between the effect of the cap and that of an infinite horizontal slab for a significant range of elevations. Figure 4 shows calculated curves, based on equation (1), similar to Figure 2 for spherical caps whose surface radii are 50, 100, 167, 200, and 250 km. The Bullard selection shows the least departure from the zero correction axis. These curves result, of course, from obvious changes in the inbalance between the truncation and subslab effects discussed above. Figure 5 is a chart showing the standard deviations from zero of the corrections for elevations from sea level to an elevation of 4000 m, for the calculations displayed in Figure 4. A minimum is achieved for the 167 km cap. A more detailed study using equation (1) (in which the surface radii are varied in increments of only one meter) shows that the Bullard radius produces a standard deviation minimum for a range of elevations from sea level to a few meters less than 4000. Thus, though somewhat arbitrary, the 167 km cap was appropriately selected and, of course, is necessarily the outer radius of the Hayford-Bowie Zone O (Swick, 1942); 166.7 km should be considered the standard distance for this parameter (LaFehr, 1991).

LATITUDE DEPENDENCE OF THE BULLARD B CORRECTION

The thickness and surface radius of the cap are, of course, independent of latitude, but the curved shape of the cap is a function of the central angle (see Appendix) and, therefore, of latitude. Although the Bullard B Correction is defined for a specific surface radius, as discussed above, it is implicitly assumed that it is computed for an earth radius at midlatitudes and that differences with respect to latitude changes should not be large. To check this assumption, we may use equation (1) to calculate the effects at the extreme latitudes and compare the results with those given in Figure 2. Figure 6 shows that only at high elevations would we encounter discrepancies of possible importance in high-precision surveys, but equation (1) can be used to accept the earth radius appropriate to the station latitude, if the investigator feels this to be important.

APPROXIMATIONS TO THE BULLARD B CORRECTION

As mentioned above, the U.S. Geological Survey (Oliver, 1980) developed a power-series approximation for the Bullard B correction. It is of the form

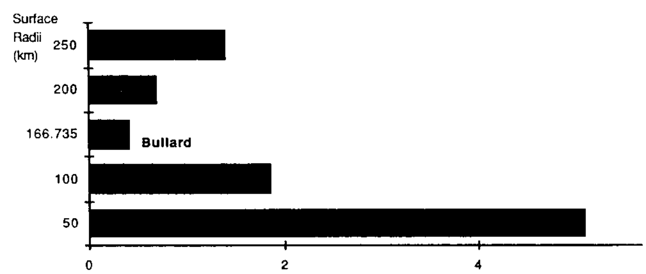

FIG. 5. Standard deviation of curvature corrections with respect to a zero correction for caps of different surface radii (mGal).

$$BB \approx Ah + Bh^2 + Ch^3.$$

Their coefficients appear to have changed slightly, depending on choice of elevation units and significant figures (Plouff, 1990, personal communication) over time, but those producing the least error are used in this discussion. Whitman (1990, personal communication) has developed an approximation derived from equation (1) of this paper in terms of actual earth parameters:

$$BB \approx 2\pi\gamma\rho h\{\alpha/2 - \eta[1 + 1/(2\alpha)]\}$$

(see appendix for definition of terms). Whitman's approximation can also be cast as a power series, the coefficients of which may then be compared with those of the U.S. Geological Survey. I have calculated a third set of power-series coefficients (solely to minimize the error), and the plot of the three approximations discussed in this paragraph are shown in Figure 7. The coefficients used in this calculation are

Coefficient	USGS	LaFehr
$A \times 10^{-3}$	1.464	1.46308
$B \times 10^{-7}$	3.533	3.52725
$C \times 10^{-14}$	4.5	5.1

Each of the three approximations may be improved by adding terms. However, the exact formula (equation 1) when put into working form (see Appendix) is very efficient and not at all difficult to implement in standard data processing routines. Personal computers and hand-held calculators provide more than the needed eight-place precision without resorting to any special programming. These approximations are shown here for comparative and historical purposes, but are not actually needed. Whitman's formula (especially in expanded form) indicates the physical basis for contributions to the correction, but all relevant physical understanding may be gained by Figures 1 and 2 and the associated discussion at the beginning of this paper.

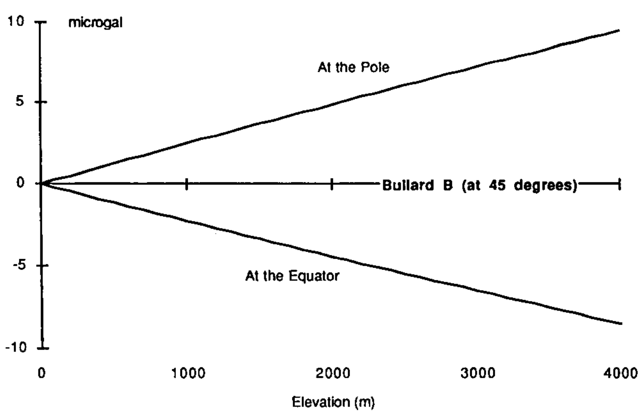

FIG. 6. Effect of latitude on the curvature correction.

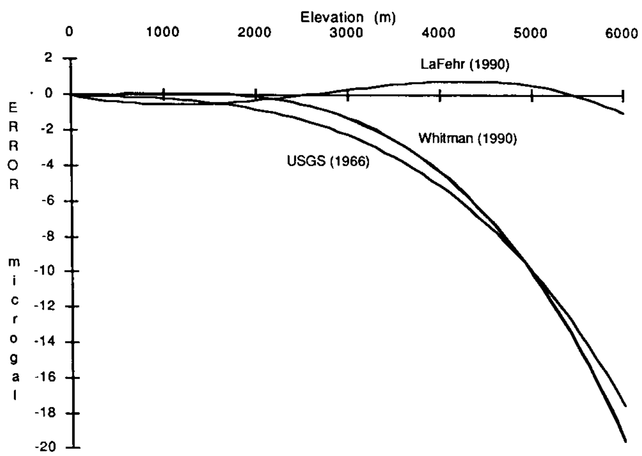

FIG. 7. Errors as a function of elevation for three approximations.

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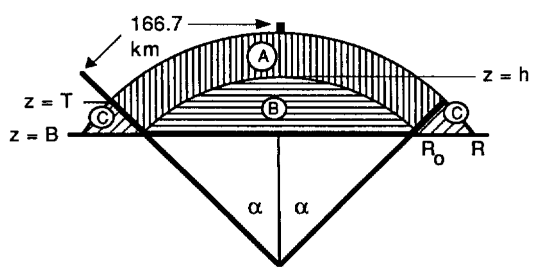
APPENDIX

CLOSED-FORM ATTRACTION FOR THE SPHERICAL CAP

In Figure A-1, the spherical earth near the station is divided into three parts: A is the spherical cap for which we wish to know the attraction on its surface at its center. B is a spherical segment of the earth whose radius is the normal mean sea level and whose base is fixed by the surface distance of the spherical cap (in the Bullard B case, this is 166.7 km). C is the nearly triangular (in cross-section) shaped ring which fills out the spherical cap to a spherical segment of the earth whose top is the earth's surface (having a radius of $R = R_0 + h$, where R_0 is the normal earth radius to mean sea level and h is the elevation of the station) and whose bottom is the same as the bottom to B. This base has a vertical displacement from the station (which we take to be the origin) of B. The attraction of the spherical cap is the sum of the three zones (the first integral, below) less the effects of zones B and C, which are represented by the second and third integrals, respectively. As shown in Figure A-1, T is the vertical distance between the station and the place where the earth cone intersects with the outer surface radius of Zone O in the Hayford-Bowie system (also the Bullard B arc length). Of course z is the variable vertical distance measured downward from the station, and dz is the differential thickness of each circular disk lying within the shaded areas of Figure A-1; γ is the universal gravity constant, ρ the density of the topography, and 2α the angle subtended at the earth's center by the spherical cap.

$$g_{sc} = 2\pi\gamma\rho \left\{ \begin{aligned} &\int_0^B \left(1 - \frac{z}{(z^2 + R^2 - (R-z)^2)^{1/2}}\right) dz - \int_h^B \left(1 - \frac{z}{(z^2 + R_0^2 - (R-z)^2)^{1/2}}\right) dz \\ &- \int_T^B \left(\frac{z}{(z^2 + (R-z)^2 \tan^2 \alpha)^{1/2}} - \frac{z}{(z^2 + R^2 - (R-z)^2)^{1/2}}\right) dz \end{aligned} \right\}.$$

Integrating and substituting the limits of integration, we have



■ is the location of the station at which z = 0 for purposes of the derivation.

R_0 is the earth's normal radius to sea level.

R is the earth's radius to the station.

h is the elevation of the station, but measured from the station to R_0 (sea level).

B is the vertical distance measured from the station to the horizontal base plane.

T is the vertical distance measured from the station to the horizontal plane which passes through the top point of Ⓒ.

α is the half angle subtended at the earth's center by the section of the earth's surface at sea level for which the outer distance from the station is normally taken to be 166.7 km (or the outer radius of the Hayford-Bowie Zone O).

Ⓑ is the shaded area which represents the first part of the total spherical sector to be removed in the curvature calculation.

Ⓒ is the shaded area which represents the second part of the total spherical sector to be removed in the curvature calculation.

FIG. A-1. Cross-section of a spherical cap indicated by the shaded area A.

$$g_{sc} = 2\pi\gamma\rho \left\{ B - \frac{1}{6R^2}\sqrt{(2RB)^3} \right.$$

$$- \left[B + \frac{2(2(R_0^2 - R^2) - 2RB)}{3(2R)^2}\sqrt{R_0^2 - R^2 + 2RB} - \left(h + \frac{2(2(R_0^2 - R^2) - 2Rh)}{3(2R)^2}\sqrt{R_0^2 - R^2 + 2Rh} \right) \right]$$

$$- \left\{ \begin{array}{l} \dfrac{\sqrt{a + bB + cB^2}}{c} - \dfrac{b}{2c}\left[\dfrac{1}{\sqrt{c}} \log_e \left(\sqrt{a + bB + cB^2} + B\sqrt{c} + \dfrac{b}{2\sqrt{c}} \right) \right] \\ - \dfrac{\sqrt{a + bT + cT^2}}{c} + \dfrac{b}{2c}\left[\dfrac{1}{\sqrt{c}} \log_e \left(\sqrt{a + bT + cT^2} + T\sqrt{c} + \dfrac{b}{2\sqrt{c}} \right) \right] \end{array} \right\} - \left\{ \begin{array}{l} \dfrac{1}{6R^2}\sqrt{(2RT)^3} \\ - \dfrac{1}{6R^2}\sqrt{(2RB)^3} \end{array} \right\} \Bigg\},$$

where $a = R^2 \tan^2 \alpha$, $b = -2R \tan^2 \alpha$, and $c = 1 + \tan^2 \alpha$. The limits and input parameters for the calculations are:

$$T = R - R\cos\alpha, \qquad B = R - R_0 \cos\alpha, \qquad \alpha = S/R_0, \qquad S = \text{Bullard B surface radius.}$$

By noting that the second terms in the first and the last integrals are the same, allowing for the canceling of appropriate terms, and simplifying, we now have a solution in closed form for the spherical cap:

$$g_{sc} = 2\pi\gamma\rho \left\{ h\left(1 - \eta + \frac{\eta^2}{3}\right) - \frac{R}{3} \left[\begin{array}{l} (\delta^2 - 2 + \delta\cos\alpha + 3\cos^2\alpha)\sqrt{(\cos\alpha - \delta)^2 + \sin^2\alpha} \\ - 6\cos^2\alpha \sin(\alpha/2) + 4\sin^3(\alpha/2) \\ - 3\sin^2\alpha\cos\alpha \log_e \dfrac{2(\sin(\alpha/2) - \sin^2(\alpha/2))}{\cos\alpha - \delta + \sqrt{(\cos\alpha - \delta)^2 + \sin^2\alpha}} \end{array} \right] \right\}, \qquad \text{(A-1)}$$

where $\delta = R_0/R$ and $\eta = h/R$.

We note that for fixed α and R_0, the simple working equation [equation (1), given in the paper] may be written for very rapid calculation:

Let $\mu = (1/3\eta^2 - \eta)$ and

$$\lambda = \frac{1}{3}\left\{ (d + f\delta + \delta^2)[(f - \delta)^2 + k]^{1/2} + p + m \log_e \frac{n}{f - \delta + [(f - \delta)^2 + k]^{1/2}} \right\},$$

where $d = 3\cos^2\alpha - 2$, $f = \cos\alpha$, $k = \sin^2\alpha$, $p = -6\cos^2\alpha\sin(\alpha/2) + 4\sin^3(\alpha/2)$, $m = -3\sin^2\alpha\cos\alpha$, and $n = 2[\sin(\alpha/2) - \sin^2(\alpha/2)]$.

The calculations which produced Figures 2 and 3 assumed a normal radius R_0 of 6371 km and a Bullard B surface radius S of 166.735 km. Variations with respect to latitude and with respect to the surface radius S are discussed in the paper.